Malthematics (Subsidiary)
Part II (Dated- 17/04/21)

Some examples of successive differentiation:

Example 1. If $y = \sin mx + \cos mx$, Prove that

$$y_n = m^n (1 + (-1)^n \sin 2mx)^{\frac{1}{2}}.$$

Solution:— It is given that $y = \sin mx + \cos mx$.

Therefore from the formula.

$$y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$$

$$= m^n \left\{ \sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right) \right\}$$

$$= m^n \left[\left\{ \sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right) \right\}^2 \right]^{\frac{1}{2}}$$

$$= m^n \left[\sin^2\left(mx + \frac{n\pi}{2}\right) + \cos^2\left(mx + \frac{n\pi}{2}\right) + 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right) \right]^{1/2}$$

$$= m^n \left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right) \right]^{\frac{1}{2}}$$

$$= m^n \left(1 + \sin(2mx + n\pi) \right)^{\frac{1}{2}}$$

$$= m^n \left[1 + (\sin 2mx \cos n\pi + \cos 2mx \sin n\pi) \right]^{\frac{1}{2}}$$

$$= m^n \left[1 + \cos n\pi \cdot \sin 2mx \right]^{\frac{1}{2}} \quad (1) \text{ Since } \sin n\pi = 0$$

Now $\cos n\pi = -1$, ~~iff~~ if n is an odd integer

and $\cos n\pi = +1$ if n is an even integer.

Therefore we can write

$$\cos n\pi = (-1)^n$$

hence from (1) $y_n = m^n (1 + (-1)^n \sin 2mx)^{\frac{1}{2}}$.

Example – If $ax^2 + 2bxy + by^2 = 1$, Proove that

$$\frac{d^2y}{dx^2} = \frac{b^2 - ab}{(bx+by)^3}$$

Solution: — Given $ax^2 + 2bxy + by^2 = 1$

Differentiating with respect to x, We get

$$a\,2x + 2b\left(x\frac{dy}{dx} + y\right) + b \cdot 2y\frac{dy}{dx} = 0$$

$$\Rightarrow ax + \frac{dy}{dx}(bx+by) + by = 0$$

$$\Rightarrow (ax+by) + \frac{dy}{dx}(bx+by) = 0$$

$$\therefore \frac{dy}{dx} = -\frac{ax+by}{bx+by}$$

Again differentiating with respect to x, We get

$$\frac{d^2y}{dx^2} = -\left[\frac{(bx+by)(a+by_1) - (ax+by)(b+by_1)}{(bx+by)^2}\right]$$

Where $y_1 = \frac{dy}{dx}$.

$$\cancel{= -(abx + b^2xy_1 + aby + bby_1y) - (abx + abxy_1}$$

$$= -\frac{(abx + b^2xy_1 + aby + bbyy_1) - (abx + abxy_1 + b^2y + bbyy_1)}{(bx+by)^2}$$

$$= -\frac{(b^2-ab)xy_1 + (ab-b^2)y}{(bx+by)^2}$$

$$= -\frac{(b^2-ab)(xy_1-y)}{(bx+by)^2} = \frac{(b^2-ab)(y-xy_1)}{(bx+by)^2} \quad \text{—— (1)}$$

But $y - xy_1 = y - x \cdot x\left(-\frac{ax+by}{bx+by}\right) = y + \frac{ax^2+bxy}{bx+by}$

$$= \frac{bxy + by^2 + ax^2 + bxy}{bx+by} = \frac{ax^2 + 2bxy + by^2}{bx+by} = \frac{1}{bx+by}$$ since $ax^2 + 2bxy + by^2 = 1$

therefore from (1) $\frac{d^2y}{dx^2} = \frac{b^2-ab}{(bx+by)^2} \times \frac{1}{(bx+by)} = \frac{b^2-ab}{(bx+by)^3}$ Ans.